

A MATROIDAL GENERALIZATION OF RESULTS OF DRISKO AND CHAPPELL

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ABSTRACT. Let \mathcal{M} and \mathcal{N} be two matroids on the same ground set. We generalize results of Drisko and Chapell by showing that any $2n - 1$ sets of size n in $\mathcal{M} \cap \mathcal{N}$ have a rainbow set of size n in $\mathcal{M} \cap \mathcal{N}$.

1. INTRODUCTION

An $m \times n$ row-Latin rectangle is an $m \times n$ array in which each of the numbers $1, \dots, n$ appear exactly once in each row. A *partial transversal of size k* in an $m \times n$ row-Latin rectangle R is a set of k entries of R such that no two of them are in the same row or in the same column. If all the elements in a partial transversal are distinct we call it a *partial rainbow transversal*.

Drisko [3] proved the following elegant result:

Theorem 1.1. *Any $(2n - 1) \times n$ row-Latin rectangle has a partial rainbow transversal of size n .*

In the same paper Drisko also gave an example of a $(2n - 2) \times n$ row-Latin rectangle in which there is no partial rainbow transversal of size n .

Given sets F_1, \dots, F_m , a *partial rainbow function* is a partial choice function of the sets F_i . A *partial rainbow set* is the range of a partial rainbow function. If the sets F_1, \dots, F_m are matchings in a given bipartite graph, then a *partial rainbow matching* is a partial rainbow set which is also a matching. Drisko's result asserts that any $2n - 1$ matchings of size n in a bipartite graph with $2n$ vertices have a partial rainbow matching of size n .

In this paper we generalize Drisko's theorem to matroids.

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We provide some definitions and notation concerning matroids. For a set A and an element x we use the notation $A + x$ for $A \cup \{x\}$ and $A - x$ for $A \setminus \{x\}$. A collection \mathcal{M} of subsets of a *ground set* \mathcal{S} is a *matroid* if it is hereditary and it satisfies an augmentation property: If $A, B \in \mathcal{M}$ and $|B| > |A|$, then there exists $x \in B \setminus A$ such that $A + x \in \mathcal{M}$. Sets in \mathcal{M} are called *independent* and subsets of \mathcal{S} not belonging to \mathcal{M} are called *dependent*. An element $x \in \mathcal{S}$ is *spanned* by A if either $x \in A$ or $I + x \notin \mathcal{M}$ for some independent set $I \subseteq A$. The set of elements that are spanned in \mathcal{M} by A is denoted by $\text{sp}_{\mathcal{M}}(A)$. A *circuit* is a minimal dependent set. For more background on matroid theory the reader is referred to in the books of Oxley [4] and Welsh [5].

Chappell [2] proved the following generalization of Drisko's theorem:

Theorem 1.2. *Any $(2n - 1) \times n$ array whose entries are taken from the ground set of a matroid \mathcal{M} , and whose rows are independent sets in \mathcal{M} , has a partial transversal of size n which is an independent set of \mathcal{M} .*

In this paper we prove the following:

Theorem 1.3. *Let \mathcal{M} and \mathcal{N} be two matroids on the same ground set \mathcal{S} . Any $2n - 1$ sets of size n , each in $\mathcal{M} \cap \mathcal{N}$, have a partial rainbow set of size n in $\mathcal{M} \cap \mathcal{N}$.*

Note that Theorem 1.1 is obtained from Theorem 1.3 by taking both \mathcal{M} and \mathcal{N} to be partition matroids, and Theorem 1.2 follows in the case that one of the matroids is a partition matroid.

2. PROOF OF THEOREM 1.3

We shall use the following basic facts on matroids:

Fact 2.1. *If $I \in \mathcal{M}$ and $I + x \notin \mathcal{M}$, there exists a unique minimal subset of I , which we denote by $C_{\mathcal{M}}(I, x)$, that spans x , and for each $a \in C_{\mathcal{M}}(I, x)$, we have $I + x - a \in \mathcal{M}$ and $\text{sp}_{\mathcal{M}}(I + x - a) = \text{sp}_{\mathcal{M}}(I)$.*

Fact 2.2 is an immediate consequence of the augmentation property:

Fact 2.2. *Let I and J be independent sets in \mathcal{M} . If $|I| < |J|$, then there exists $J_1 \subseteq J \setminus I$ such that $|J_1| \geq |J| - |I|$ and $I \cup J_1 \in \mathcal{M}$.*

Fact 2.3 is also known as the *circuit elimination axiom*:

Fact 2.3. *If C_1 and C_2 are circuits with $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ then there exists a circuit C_3 such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$.*

We shall also need the following lemma from [1]. The proof is repeated here in order to make the discussion self-contained.

Lemma 2.4. *Let \mathcal{M} be a matroid. Let $I \in \mathcal{M}$ and $X = \{x_1, \dots, x_k\} \subseteq I$ and $Y = \{y_1, \dots, y_k\} \subseteq \text{sp}_{\mathcal{M}}(I) \setminus I$ be such that $\text{sp}_{\mathcal{M}}((I \setminus X) \cup Y) = \text{sp}_{\mathcal{M}}(I)$. Suppose $y_{k+1} \in \text{sp}_{\mathcal{M}}(I) \setminus I$ and x_{k+1} are such that $x_{k+1} \in C_{\mathcal{M}}(I, y_{k+1}) \setminus X$ and $x_{k+1} \notin C_{\mathcal{M}}(I, y_i)$ for all $i = 1, \dots, k$. Then $x_{k+1} \in C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1})$.*

Proof. Suppose, for contradiction, that $x_{k+1} \notin C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1})$. Let $C_1 = C_{\mathcal{M}}(I, y_{k+1}) + y_{k+1}$ and $C_2 = C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1}) + y_{k+1}$. Then, by Fact 2.3, there exists a circuit $C^1 \subseteq C_1 \cup C_2$, such that $x_{k+1} \in C^1$ and $y_{k+1} \notin C^1$. Since I is independent, C^1 must contain at least one element $y_j \in Y$. Let $C_3 = C_{\mathcal{M}}(I, y_j) + y_j$. Since $x_{k+1} \notin C_{\mathcal{M}}(I, y_j)$, there exists a circuit $C^2 \subseteq C^1 \cup C_3$ such that $x_{k+1} \in C^2$ and $y_j \notin C^2$, by Fact 2.3. Since $C^2 \cap Y \subset C^1 \cap Y$ we must have $|C^2 \cap Y| < |C^1 \cap Y|$. We proceed this way until we obtain a circuit whose intersection with Y is empty. This will contradict the independence of I . \square

Proof of Theorem 1.3. Let $A_1, \dots, A_{2n-1} \in \mathcal{M} \cap \mathcal{N}$, each of size n . Let $R \in \mathcal{M} \cap \mathcal{N}$ be a partial rainbow set of maximal size. Assume, for contradiction, that $|R| < n$. Without loss of generality we may assume that $R \cap A_i = \emptyset$ for $i = 1, \dots, n$. We define,

Definition 2.5. A *colorful alternating trail* (CAT) of length k ($1 \leq k \leq n-1$) relative to R consists of a set $\{a_1, \dots, a_k\}$, where distinct a_i 's belong to distinct A_j 's ($j \in \{1, \dots, n-1\}$) and a set $\{r_1, \dots, r_k\} \subset R$, such that

- (P_M) $R + a_1 - r_1 + a_2 - r_2 + \dots - r_{i-1} + a_i \in \mathcal{M}$ for all $i = 1, \dots, k$.
- (P_N) $R + a_1 - r_1 + a_2 - r_2 + \dots + a_i - r_i \in \mathcal{N}$ and $\text{sp}_{\mathcal{N}}(R + a_1 - r_1 + a_2 - r_2 + \dots + a_i - r_i) = \text{sp}_{\mathcal{N}}(R)$ for all $i = 1, \dots, k$.

If, in addition, $R + a_1 - r_1 + a_2 - r_2 + \dots + a_{k-1} - r_{k-1} + a_k \in \mathcal{N}$ then the CAT is called *augmenting* (in this case the condition (P_N) for $i = k$ is redundant).

Note that since R is of maximal size, no augmenting CAT relative to R exists. The theorem will be proved by showing that the assumption $|R| < n$ yields a partial rainbow set of size $|R| + 1$ in $\mathcal{M} \cap \mathcal{N}$.

Claim 1. For each $k = 1, \dots, n-1$, the CATs involving only elements of A_1, \dots, A_k contain at least k distinct elements of R .

Proof of Claim 1. We prove Claim 1 by induction on k . Since $|R| < n$ and $|A_1| = n$ there exists an element $a_1 \in A_1$ such that $R + a_1 \in \mathcal{M}$. By the maximality property of R we must have $R + a_1 \notin \mathcal{N}$. By Fact 2.1, there exists an element $r_1 \in R$ such that $R + a_1 - r_1 \in \mathcal{N}$ and $\text{sp}_{\mathcal{N}}(R + a_1 - r_1) = \text{sp}_{\mathcal{N}}(R)$. Thus, $\{a_1\}$ and $\{r_1\}$ form a CAT of length 1 and Claim 1 holds for $k = 1$.

Now suppose Claim 1 holds for $k-1$ for some $k \geq 2$. Without loss of generality we may assume that the CATs involving only elements of A_1, \dots, A_{k-1} contain the elements $r_1, \dots, r_{k-1} \in R$. Let $R_{k-1} = \{r_1, \dots, r_{k-1}\}$.

Since $|R| < n$, Fact 2.2 implies that the set A_k contains at least k elements that are not in $\text{sp}_{\mathcal{M}}(R \setminus R_{k-1})$. Since $|R_{k-1}| = k-1$, at least one of these elements is not in $\text{sp}_{\mathcal{N}}(R_{k-1})$. Let a be such an element. That is, $a \in A_k$ and

$$(2.1) \quad a \notin \text{sp}_{\mathcal{M}}(R \setminus R_{k-1})$$

and

$$(2.2) \quad a \notin \text{sp}_{\mathcal{N}}(R_{k-1}).$$

First assume $a \notin \text{sp}_{\mathcal{M}}(R)$. Then $R + a \in \mathcal{M}$. Since R is of maximal size it follows that $R + a \notin \mathcal{N}$. By (2.2), $C_{\mathcal{N}}(R, a) \not\subseteq R_{k-1}$ and thus, there exists an element

$r \in R \setminus R_{k-1}$ such that $R + a - r \in \mathcal{N}$ and $\text{sp}_{\mathcal{N}}(R + a - r) = \text{sp}_{\mathcal{N}}(R)$. The CAT consisting of $\{a\}$ and $\{r\}$ contains the extra element $r \notin R_{k-1}$, and thus Claim 1 holds for k .

Now assume that $a \in \text{sp}_{\mathcal{M}}(R)$. By (2.1), there exists $r' \in C_{\mathcal{M}}(R, a) \cap R_{k-1}$. By the definition of R_{k-1} , there exists a CAT containing r' , say

$$(2.3) \quad \{a_{i_1}, \dots, a_{i_l}\} \text{ and } \{r_{i_1}, \dots, r_{i_l} = r'\}.$$

We may assume that none of $r_{i_1}, \dots, r_{i_{l-1}}$ is in $C_{\mathcal{M}}(R, a)$ (otherwise we take r' to be the first of $r_{i_1}, \dots, r_{i_{l-1}}$ that belongs to $C_{\mathcal{M}}(R, a)$). Let $R' = R + a_{i_1} - r_{i_1} + \dots - r_{i_{l-1}} + a_{i_l}$. Since $R' \in \mathcal{M}$ and no element of $C_{\mathcal{M}}(R, a)$ was discarded along the trail from R to R' , we have $C_{\mathcal{M}}(R', a) = C_{\mathcal{M}}(R, a)$. Hence, $r' \in C_{\mathcal{M}}(R', a)$. By Fact 2.1 we have

$$(2.4) \quad R' - r' + a \in \mathcal{M}.$$

If $a \notin \text{sp}_{\mathcal{N}}(R)$ then $R + a \in \mathcal{N}$. By Property $(P_{\mathcal{N}})$ we have $\text{sp}_{\mathcal{N}}(R' - r') = \text{sp}_{\mathcal{N}}(R)$. Thus, $R' - r' + a \in \mathcal{N}$, which, together with (2.4), yields an augmenting CAT, contrary to the maximality of $|R|$. Hence, we may assume that $a \in \text{sp}_{\mathcal{N}}(R)$. By (2.2), there exists an element $r'' \in C_{\mathcal{N}}(R, a) \setminus R_{k-1}$.

If none of the a_{i_j} ($j = 1, \dots, l$) in (2.3) satisfies $r'' \in C_{\mathcal{N}}(R, a_{i_j})$, then, by Lemma 2.4, $r'' \in C_{\mathcal{N}}(R' - r', a)$, and thus, the sets $\{a_{i_1}, \dots, a_{i_l}, a\}$ and $\{r_{i_1}, \dots, r_{i_l}, r', r''\}$ make a CAT which contains the extra element $r'' \notin R_{k-1}$, proving Claim 1 for k .

Otherwise, let j be minimal such that $r'' \in C_{\mathcal{N}}(R, a_{i_j})$ and let $R'' = R + a_{i_1} - r_{i_1} + \dots + a_{i_{j-1}} - r_{i_{j-1}}$. Then, by Lemma 2.4, it follows that $r'' \in C_{\mathcal{N}}(R'', a_{i_j})$. Hence, $R'' + a_{i_j} - r'' \in \mathcal{N}$ and $\text{sp}_{\mathcal{N}}(R'' + a_{i_j} - r'') = \text{sp}_{\mathcal{N}}(R'') = \text{sp}_{\mathcal{N}}(R)$. We obtain a CAT consisting of $\{a_{i_1}, \dots, a_{i_{j-1}}, a_{i_j}\}$ and $\{r_{i_1}, \dots, r_{i_{j-1}}, r''\}$. This CAT involves the extra element $r'' \notin R_{k-1}$. Thus, the CATs involving only elements of A_1, \dots, A_{k-1}, A_k contain at least k elements of R , proving Claim 1 for k . This completes the proof of Claim 1.

To conclude the proof of Theorem 1.3, note that by Claim 1, the CATs involving only elements of A_1, \dots, A_{n-1} contain all the elements of R . Since $|R| < n$ there exists an element $a_n \in A_n$ such that $R + a_n \in \mathcal{N}$. By the maximality property of R , we must have that $a_n \in \text{sp}_{\mathcal{M}}(R)$. Let $r \in R$ satisfy $R + a_n - r \in \mathcal{M}$. Applying Claim 1 to the case $k = n - 1$, it follows that r belongs to a CAT consisting of some sets $\{a_{i_1}, \dots, a_{i_l}\}$ and $\{r_{i_1}, \dots, r_{i_{l-1}}, r_{i_l} = r\}$, where each of a_{i_1}, \dots, a_{i_l} belongs to a different set among A_1, \dots, A_{n-1} . Let $R' = R + a_{i_1} - r_{i_1} + \dots - r_{i_{l-1}} + a_{i_l}$. We may assume that r is the first element in this trail satisfying $r \in C_{\mathcal{M}}(R, a_n)$ (otherwise we take r to be the first element in the trail belonging to $C_{\mathcal{M}}(R, a_n)$) and thus, $C_{\mathcal{M}}(R', a_n) = C_{\mathcal{M}}(R, a_n)$. Hence, $R' - r + a_n \in \mathcal{M}$. Since $\text{sp}_{\mathcal{N}}(R' - r) = \text{sp}_{\mathcal{N}}(R)$, by Property $(P_{\mathcal{N}})$, we also have $R' - r + a_n \in \mathcal{N}$. Thus, $R' - r + a_n$ is a rainbow matching of size $|R| + 1$. \square

REFERENCES

- [1] R. Aharoni, D. Kotlar, and R. Ziv, *Rainbow sets in the intersection of two matroids*, a manuscript [arXiv:1405.3119 \[math.CO\]](#).

- [2] G. G. Chappell, *A matroid generalization of a result on row-latin rectangles*, Journal of Combinatorial Theory, Series A **88** (1999), no. 2, 235–245.
- [3] A. A. Drisko, *Transversals in row-Latin rectangles*, Journal of Combinatorial Theory, Series A **84** (1998), 181–195.
- [4] J. Oxley, *Matroid theory*, 2 ed., Oxford University Press, 2011.
- [5] D. Welsh, *Matroid theory*, Academic Press, London, 1976.